C4 Exercise 6C (integration using trig. identities)

examples

find $\int \tan^2 x \, dx$

since $\sec^2 x \equiv 1 + \tan^2 x$

then $\tan^2 x \equiv \sec^2 x - 1$

Hence $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C$

C4 Ex 6C

examples

find $\int \sin^2 x \, dx$

Since $\cos 2x \equiv \cos^2 x - \sin^2 x \equiv (1 - \sin^2 x) - \sin^2 x$

then $\cos 2x \equiv 1 - 2\sin^2 x$

so $2\sin^2 x \equiv 1 - \cos 2x$

and hence $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$

then $\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

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examples

find $\int \sin 3x \cos 3x \, dx$

now since $\sin 2\theta = 2\sin\theta\cos\theta$

we have $\sin 3x \cos 3x = \frac{1}{2}\sin 2(3x) = \frac{1}{2}\sin 6x$

so $\int \sin 3x \cos 3x \, dx = \int \frac{1}{2}\sin 6x \, dx = \frac{-1}{12}\cos 6x + C$

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examples

find $\int (\sec x + \tan x)^2 \, dx$

$$= \int \left(\sec^2 x + 2\sec x \tan x + \tan^2 x\right) dx$$

Now $\int \sec^2 x = \tan x + C$

$\int 2\sec x \tan x = 2\sec x + C$

*see first example above

and $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C$

$\therefore \int (\sec x + \tan x)^2 \, dx = 2\tan x - x + 2\sec x + C$

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example

find $\int \sin 3x \cos 2x \, dx$

⚠ sneaky trick: $\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

while $\sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

adding get: $\sin(5x) + \sin(x) = 2 \sin 3x \cos 2x$

so $\int \sin 3x \cos 2x \, dx = \int \frac{1}{2}(\sin 5x + \sin x) \, dx$

$= -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C$

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example :

This clearly will generalise to any expression of the form

$$\sin Ax \cos Bx$$

Then $\sin(Ax + Bx) \equiv \sin Ax \cos Bx + \cos Ax \sin Bx$

and $\sin(Ax - Bx) \equiv \sin Ax \cos Bx - \cos Ax \sin Bx$

$$\Rightarrow 2\sin Ax \cos Bx \equiv \sin[(A+B)x] + \sin[(A-B)x]$$

$$\Rightarrow \sin Ax \cos Bx \equiv \frac{1}{2}\left\{ \sin[(A+B)x] + \sin[(A-B)x] \right\}$$

C4Ex6C

1a integrate $\cot^2 x$

$$\cot^2 x \equiv \operatorname{cosec}^2 x - 1$$

$$\Rightarrow \int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\cot x - x + C$$

C4Ex6C

1b integrate $\cos^2 x$

use $\cos 2x \equiv 2\cos^2 x - 1$

$\Rightarrow \quad \cos^2 x \equiv \frac{1}{2}(\cos 2x - 1)$

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2}\cos 2x - \frac{1}{2}\right) dx = \frac{1}{4}\sin 2x - \frac{1}{2}x + C$$

C4 Ex 6C

1c $\int \sin 2x \cos 2x \, dx = \int \frac{1}{2}\left(\sin 4x - \sin 0\right) dx$ ★

★ see page 6 of this document

since $\sin(2x + 2x) = \sin 2x \cos 2x + \sin 2x \cos 2x$

$\Rightarrow \sin 4x = 2 \sin 2x \cos 2x.$

Hence

$$\int \sin 2x \cos 2x \, dx = \int \frac{1}{2} \sin 4x \, dx$$

$$= -\frac{1}{8} \cos 4x + C$$

C4Ex6C

1d $\int (1 + \sin x)^2 \, dx = \int (1 + 2\sin x + \sin^2 x) \, dx$

$$\cos 2x \equiv \cos^2 x - \sin^2 x \equiv 1 - 2\sin^2 x$$

$$\Rightarrow \quad 2\sin^2 x \equiv 1 - \cos 2x$$

$$\Rightarrow \quad \sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\Rightarrow \int (1 + \sin x)^2 \, dx = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{3x}{2} - 2\cos x - \frac{1}{4}\sin 2x + C$$

C4 Ex 6C

1e find $\int \tan^2 3x \, dx$

Now since $\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$

$$\int \tan^2 3x \equiv \int (\sec^2 3x - 1) \, dx$$

$$= \frac{1}{3}\tan 3x - x + C$$

C4 Ex 6C

1f find $\int (\cot x - \operatorname{cosec} x)^2 \, dx$

$$= \int \left(\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x\right) dx$$

But since $\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x} \Rightarrow \cot^2 x + 1 \equiv \operatorname{cosec}^2 x$

then $\int (\cot x - \operatorname{cosec} x)^2 \, dx \equiv \int \left(2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x\right) dx$

$$= -2\cot x - x + 2\operatorname{cosec} x + C$$

C4Ex6C

1g find $\int (\sin x + \cos x)^2 \, dx \equiv \int \left(\sin^2 x + \cos^2 x + 2\sin x \cos x\right) dx$

$\equiv \int 1 + 2\sin x \cos x \quad dx$

Since $\sin 2x = 2 \sin x \cos x$

$\int (\sin x + \cos x)^2 \, dx \equiv \int (1 + \sin 2x) \, dx$

$= x - \frac{1}{2}\cos 2x + C$

C4 Ex6C

1h find $\int \sin^2 x \cos^2 x \, dx$

$\sin^2 x \cos^2 x \equiv \cos^2 x - \cos^4 x$

now $\cos^2 2x = (\cos 2x)^2 = (2\cos^2 x - 1)^2$

$\Rightarrow \cos^2 2x = 4\cos^4 x - 4\cos^2 x + 1$

$\Rightarrow 4(\cos^2 x - \cos^4 x) \equiv 1 - \cos^2 2x$

$\Rightarrow \cos^2 x - \cos^4 x \equiv \frac{1}{4} - \frac{1}{4}\cos^2 2x$ ☆

now $\cos 2\theta \equiv 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

replace $\theta \equiv 2x$ in ☆

$\Rightarrow \cos^2 x - \cos^4 x \equiv \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)$

So in $\int \sin^2 x \cos^2 x \, dx \equiv \int \left(\cos^2 x - \cos^4 x\right) dx$

replace this by $\equiv \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

C4Ex6C

1i $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \frac{1}{(\sin x \cos x)^2}\,dx$

now $\sin 2x = 2\sin x \cos x$

so $\sin x \cos x = \frac{1}{2}\sin 2x$

hence $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \frac{1}{\frac{1}{4}\sin^2 2x}\,dx = \int 4\,\mathrm{cosec}^2 2x\,dx$

$$= -\frac{4}{2}\cot 2x + C$$

$$= -2\cot 2x + C$$

Or you might use the following method instead...

C4Ex6C

1i find $\int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \operatorname{cosec}^2 x \sec^2 x\,dx$

$$\frac{\sin^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} \Rightarrow 1 + \tan^2 x \equiv \sec^2 x$$

$$\frac{\cos^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} \Rightarrow 1 + \cot^2 x \equiv \operatorname{cosec}^2 x$$

$$\Rightarrow \int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int (1+\cot^2 x)(1+\tan^2 x)\,dx$$

$$= \int (1 + \cot^2 x + \tan^2 x + 1)\,dx$$

$$= \int \{2 + (\operatorname{cosec}^2 x - 1) + (\sec^2 x - 1)\}\,dx$$

$$\Rightarrow \int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \left(\operatorname{cosec}^2 x + \sec^2 x\right) dx$$

$$= -\cot x + \tan x + C$$

in the back it gives the solution $-2\cot 2x + C$
are these equal?

$$\cot 2x \equiv \frac{\cos 2x}{\sin 2x} \equiv \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x}$$

$$\cot 2x \equiv \frac{1}{2}\cot x - \frac{1}{2}\tan x$$

$$\Rightarrow -2\cot 2x = -\cot x + \tan x$$

$\therefore$ solution identical.

1j find $\int (\cos 2x - 1)^2 \, dx = \int \left(\cos^2 2x + 1 - 2\cos 2x\right) dx$

$$\cos 4x \equiv \cos^2 2x - \sin^2 2x$$

$$\cos 4x \equiv 2\cos^2 2x - 1$$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{2}\cos 4x \equiv \cos^2 2x$$

$$\Rightarrow \quad \int (\cos 2x - 1)^2 \, dx \equiv \int \left[\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) + \left(1 - 2\cos 2x\right)\right] dx$$

$$\equiv \int \left[\frac{3}{2} - \frac{1}{2}\cos 4x - 2\cos 2x\right] dx$$

$$= \frac{3}{2}x - \frac{1}{8}\sin 4x - \sin 2x + C$$

C4Ex6C

2a find $\int \left(\frac{1 - \sin x}{\cos^2 x} \right) dx \equiv \int (\sec^2 x - \tan x \sec x)\, dx$

$= \tan x - \sec x + C$

C4 Ex6C

2b $\displaystyle\int \frac{1 + \cos x}{\sin^2 x}\, dx \equiv \int \left(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x\right) dx$

$= \cot x + \operatorname{cosec} x + C$

C4 Ex 6C

2c $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$

$$\cos 2x \equiv 2\cos^2 x - 1$$

so $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx \equiv \int \left(\frac{2\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$

$$\equiv \int (2 - \sec^2 x)\,dx$$

$$= 2x - \tan x + C$$

C4 Ex 6C

2d

$$\int \frac{\cos^2 x}{\sin^2 x}\,dx \equiv \int \cot^2 x \,dx$$

$$\equiv \int (\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\cot x - x + C$$

C4 Ex 6C

2e

$$\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx \equiv \int \frac{(1+2\cos x+\cos^2 x)}{\sin^2 x}\,dx$$

$$\equiv \int \left(\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x\right) dx$$

$$\equiv \int \left[\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + (\operatorname{cosec}^2 x - 1)\right] dx$$

$$\equiv \int \left[2\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x - 1\right] dx$$

$$= -2\cot x - 2\operatorname{cosec} x - x + C$$

C4 Ex6C

2f

$$\int \frac{(1 + \sin x)^2}{\cos^2 x}\, dx \equiv \int \frac{(1 + 2\sin x + \sin^2 x)}{\cos^2 x}\, dx$$

$$\equiv \int \left(\frac{2 - \cos^2 x + 2\sin x}{\cos^2 x} \right) dx$$

$$\equiv \int \left[2\sec^2 x - 1 + 2\tan x \sec x \right] dx$$

$$= 2\tan x - x + 2\sec x + C$$

C4 Ex 6C

2g

$$\int (\cot x - \tan x)^2 \, dx \equiv \int (\cot^2 x - 2\cot x \tan x + \tan^2 x) \, dx$$

$$\equiv \int [(\operatorname{cosec}^2 x - 1) - 2 + (\sec^2 x - 1)] \, dx$$

$$\equiv \int (\operatorname{cosec}^2 x + \sec^2 x - 4) \, dx$$

$$= -\cot x + \tan x - 4x + C$$

C4Ex6C

2h

$$\int (\cos x - \sin x)^2 \, dx \equiv \int (\cos^2 x - 2\sin x \cos x + \sin^2 x) \, dx$$

$$\equiv \int (1 - 2\sin x \cos x) \, dx$$

Now $\sin 2x \equiv 2\sin x \cos x$

$$\Rightarrow \int (\cos x - \sin x)^2 \, dx \equiv \int (1 - \sin 2x) \, dx$$

$$= x + \frac{1}{2}\cos 2x + C$$

C4 Ex6C

2i

$$\int (\cos x - \sec x)^2 \, dx \equiv \int \left(\cos^2 x - 2\cos x \sec x + \sec^2 x\right) dx$$

$$\equiv \int \left(\cos^2 x + \sec^2 x - 2\right) dx$$

Now $\cos 2x \equiv 2\cos^2 x - 1$

$$\Rightarrow \quad \cos^2 x \equiv \frac{1}{2} + \frac{1}{2}\cos 2x$$

So $\int (\cos x - \sec x)^2 \, dx \equiv \int \left(\frac{1}{2}\cos 2x + \sec^2 x - \frac{3}{2}\right) dx$

$$= \frac{1}{4}\sin 2x + \tan x - \frac{3}{2}x + C$$

C4Ex6C

2j

$$\int \frac{\cos 2x}{1 - \cos^2 2x} \, dx \equiv \int \frac{\cos 2x}{\sin^2 2x} \, dx$$

$$\equiv \int \cot 2x \operatorname{cosec} 2x \, dx$$

$$\equiv -\frac{1}{2} \operatorname{cosec} 2x + C$$

C4 Ex 6C

3a $\int \cos 2x \cos x \, dx$

Since: $\cos(2x + x) \equiv \cos 2x \cos x - \sin 2x \sin x$

and: $\cos(2x - x) \equiv \cos 2x \cos x + \sin 2x \sin x$

adding we get: $\cos(3x) + \cos(x) \equiv 2 \cos 2x \cos x$

$\Rightarrow \quad \cos 2x \cos x \equiv \frac{1}{2} \cos 3x + \frac{1}{2} \cos x$

so $\int \cos 2x \cos x \, dx \equiv \int \left(\frac{1}{2} \cos 3x + \frac{1}{2} \cos x\right) dx$

$= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C$

C4 Ex6C

3b $\int 2\sin 5x \cos 3x \, dx$

Since $\sin(5x + 3x) \equiv \sin 5x \cos 3x + \cos 5x \sin 3x$

and $\sin(5x - 3x) \equiv \sin 5x \cos 3x - \cos 5x \sin 3x$

adding we get $\sin(8x) + \sin(2x) \equiv 2\sin 5x \cos 3x$

so $\int 2\sin 5x \cos 3x \, dx \equiv \int (\sin 8x + \sin 2x) \, dx$

$= -\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x + C$

C4 Ex 6C

3c $\displaystyle\int 2\sin 3x \cos 5x \; dx$

Since $\sin(3x + 5x) \equiv \sin 3x \cos 5x + \cos 3x \sin 5x$

and $\sin(3x - 5x) \equiv \sin 3x \cos 5x - \cos 3x \sin 5x$

adding we get $\sin(8x) + \sin(-2x) \equiv 2\sin 3x \cos 5x$

so $\displaystyle\int 2\sin 3x \cos 5x \; dx \equiv \int (\sin 8x + \sin -2x) \; dx$

$$= \frac{-1}{8}\cos 8x + \frac{1}{2}\sin(-2x) + C$$

you may note that $\sin -2x \equiv -\sin 2x$ to give

$$= \frac{-1}{8}\cos 8x - \frac{1}{2}\sin 2x + C$$

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3d $\int 2\sin 2x \sin 5x \; dx$

Since $\cos(2x - 5x) \equiv \cos 2x \cos 5x + \sin 2x \sin 5x$

and $\cos(2x + 5x) \equiv \cos 2x \cos 5x - \sin 2x \sin 5x$

subtract the second from the first to derive:

$$\cos(-3x) - \cos(7x) \equiv 2\sin 2x \sin 5x$$

now since $\cos(-3x) \equiv \cos 3x$,

$$\int 2\sin 2x \sin 5x \; dx \equiv \int (\cos 3x - \cos 7x)\, dx$$

$$= \frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x + C$$

C4 Ex 6C

3e $\quad 4\int \cos 3x \cos 7x \, dx$

Since $\cos(3x + 7x) \equiv \cos 3x \cos 7x - \sin 3x \sin 7x$

and $\cos(3x - 7x) \equiv \cos 3x \cos 7x + \sin 3x \sin 7x$

adding, and replacing $\cos(-4x)$ with $\cos 4x$

$$\cos 10x + \cos 4x \equiv 2\cos 3x \cos 7x$$

$$\Rightarrow 4\int \cos 3x \cos 7x \, dx = 2\int (\cos 10x + \cos 4x)\, dx$$

$$= 2\left(\frac{1}{10}\sin 10x + \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{1}{5}\sin 10x + \frac{1}{2}\sin 4x + C$$

C4Ex6C

3f $\int 2\cos 4x \cos 4x \, dx$

now $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$\Rightarrow \cos 2\theta \equiv 2\cos^2\theta - 1$

put $\theta = 4x$ to derive

$\int (2\cos 4x \cos 4x) \, dx \equiv \int (1 + \cos 8x) \, dx$

$= x + \frac{1}{8}\sin 8x + C$

C4 Ex 6C

3g $\int 2\cos 4x \sin 4x \, dx$

$$\sin(4x + 4x) \equiv \sin 4x \cos 4x + \cos 4x \sin 4x$$

$$\Rightarrow \quad 2\cos 4x \sin 4x \equiv \sin 8x$$

so $\int 2\cos 4x \sin 4x \, dx \equiv \int \sin 8x \, dx$

$$= -\frac{1}{8} \cos 8x + C$$

C4 Ex 6C

3h $\int 2\sin 4x \sin 4x \, dx$

since $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$\Rightarrow \cos 2\theta \equiv 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta \equiv 1 - \cos 2\theta$

put $\theta = 4x$ above to get

$$\int 2\sin 4x \sin 4x \, dx \equiv \int (1 - \cos 8x) \, dx$$

$$= x - \frac{1}{8}\sin 8x + C$$